King Solomon's Dilemma
Unbounded Values

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Here we have two agents (mothers), 1 and 2, and a prize. Define the set of possible individual private values for the prize as $V = \left\{(v_1, v_2) \in \mathbb{R}^2 : 0 \leq v_2 < v_1\right\}$ so that agent 1 has strictly greater private value than agent 2. Let $f(v_1, v_2)$ be any positive common prior density on V which is bounded above in some neighborhood of $(0, 0)$.[2] Each agent will know his own value and whether he is 'agent 1' or 'agent 2' but not the value of the other agent. From the story of King Solomon, we note here that the true mother has value v_1 and the impostor has value v_2. We would like to assign the prize to agent 1 with no monetary transfers from either agent.

Considering the class of (weakly) monotone continuous bid functions, the main result here is that every Bayesian-Nash Equilibrium of the auction mechanism (induced) game as defined in Perry-Reny (1999) has the impostor bidding zero. More precisely, if agent 2 is considerate of agent 1's best response, then agent 2 strictly prefers $b_2(\cdot) \equiv 0$ to all other bid functions.

The proposed mechanism is a second-price, all-pay auction with an option to quit. For a more complete definition, see Perry-Reny (1999). For tie-breaking, both pay the common bid and a coin is flipped to decide the allocation of the prize. The utility to agent i, given the allocation of the prize and the assigned payments, is $u_i = \delta_i \cdot v_i - p_i$ where v_i is i's private value, p_i is i's payment and δ_i is an allocation indicator, i.e. $\delta_i = 1$ if i gets the prize and $\delta_i = 0$ otherwise. Since we assume that the agent quits if and only if he must pay more for the prize than his private value, the set of strategies for agent i has essentially degenerated to the set of functions from i's type space, $\mathbb{R}_{++}$ for agent 1 and $\mathbb{R}_+$ for agent 2, to a bid in $\mathbb{R}_+$. Let $b_2(v_2)$ be 2's bid function, and let $b_1(v_1)$ be a best response for 1. We will use the following Claims.

Claim 1: $b_1(v_1) > 0$ for any $v_1 > 0$ and any $b_2(\cdot)$.

Claim 2: If $b_2(0) = 0$ and $b_2(v_1) > v_1$, then $v_1 \geq b_1(v_1)$.

Claim 3: No equilibrium exists such that $b_2(0) > 0$.

Claim 4: If $b_2(v_2) \leq v_2$ on $[0, v_1]$, then $b_1(v_1) \geq b_2(v_1)$.

The proofs of the Claims are left to the Appendix. Now, we want to show that any pair of continuous monotone bid functions (b_1, b_2) where $b_2(v_2) > 0$ for some v_2 cannot be an equilibrium. This is the main result.

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[2] This is to ensure that agent 2 cannot infer too much about 1's value from his own value.

***Proposition* :** If $b_1$and b_2 are weakly monotone continuous bid functions and (b_1, b_2) is a Bayesian-Nash Equilibrium, then $b_2(v_2) = 0$ for all v_2.

Proof of the Proposition: We know from Claim 3 above that we only need to consider pairs such that $b_2(0) = 0$. Given that $b_2(0) = 0$, there are only two possibilities: (i) either there exists an $\varepsilon > 0$ such that $b_2(v_2) \leq v_2$ on $[0, \varepsilon]$ or (ii) for every $\varepsilon > 0$ there is a $v_2^{\varepsilon} \in (0, \varepsilon)$ such that $b_2(v_2^{\varepsilon}) > v_2^{\varepsilon}$. Let $\underline{b_i^{-1}}(v) = \min\{x \in \mathbb{R}_+ : b_i(v) = x\}$ and $\overline{b_i^{-1}}(v) = \max\{x \in \mathbb{R}_+ : b_i(v) = x\}$. We begin by showing that there are no equilibria under possibility (ii).

Let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n > 0$, and $\varepsilon_n \searrow 0$ as $n \to \infty$. In addition to $b_2(0) = 0$, suppose for each n there is a $v_2^{\varepsilon_n} \in (0, \varepsilon_n)$ which satisfies $b_2(v_2^{\varepsilon_n}) > v_2^{\varepsilon_n}$. By continuity, there exists a δ_n *neighborhood* around $v_2^{\varepsilon_n}$ such that for all $v_2^n \in (v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n)$, $b_2(v_2^n) > v_2^n$.

By Claim 2 above, $b_1(v_2^n) \leq v_2^n$ for each n and all $v_2^n \in (v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n)$, and by Claim 1, there exists an $n^* > 0$ such that the $b_1^{-1}(v_2^n)$ and $b_1^{-1}(b_2(v_2^n))$ are well-defined for all $n > n^*$. For such $n > n^*$, agent 2's bid function must satisfy

$$\int_{v_2^n}^{\underline{b_1^{-1}}(v_2^n)} (v_2^n - b_1(v_1)) f(v_1|v_2^n)\, dv_1 + \int_{\underline{b_1^{-1}}(v_2^n)}^{\overline{b_1^{-1}}(v_2^n)} (v_2^n - b_1(v_1)) f(v_1|v_2^n)\, dv_1$$

$$\geq \int_{\underline{b_1^{-1}}(b_2(v_2^n))}^{\overline{b_1^{-1}}(b_2(v_2^n))} \cdot \frac{1}{2} (b_2(v_2^n) - v_2^n) f(v_1|v_2^n)\, dv_1 + \int_{\overline{b_1^{-1}}(b_2(v_2^n))}^{\infty} b_2(v_2^n) f(v_1|v_2^n)\, dv_1.$$

for almost every[3] $v_2^n \in (v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n)$.

The inequality says that if agent 2 bids above his value, to possibly entice agent 1 to bid below his respective value, 2's expected gain must not be less than his losses. Otherwise, he would do better bidding 0. Noting that $b_2(v_2^n) > v_2^n$, and given the definitions of $\overline{b_1^{-1}}$ and $\underline{b_1^{-1}}$, the above inequality implies that

$$\begin{aligned} \int_{v_2^n}^{\underline{b_1^{-1}}(b_2(v_2^n))} v_2^n f(v_1|v_2^n)\, dv_1 &\geq \int_{\underline{b_1^{-1}}(b_2(v_2^n))}^{\infty} v_2^n f(v_1|v_2^n)\, dv_1 \\ &\Longrightarrow \\ v_2^n \cdot F\left(\underline{b_1^{-1}}(b_2(v_2^n))\,|v_2^n\right) &\geq v_2^n \cdot \left(1 - F\left(\underline{b_1^{-1}}(b_2(v_2^n))\,|v_2^n\right)\right) \\ &\Longrightarrow \\ v_2^n \cdot \left(2 \cdot F\left(\underline{b_1^{-1}}(b_2(v_2^n))\,|v_2^n\right) - 1\right) &\geq 0. \end{aligned}$$

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[3] According to the unconditional proability measure $F(v_2)$.

Considering $v_2^n > 0$, this implies that

$$F\left(\underline{b_1}^{-1}\left(b_2\left(v_2^n\right)\right)|v_2^n\right) \geq \frac{1}{2} \text{ for each } n > n^* \text{ and almost every } v_2^n \in \left(v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n\right).$$

For each $n > n^*$, given δ defined above, let

$$\check{F}\left(\underline{b_1}^{-1}\left(b_2\left(v_2^{\varepsilon_n}\right)\right)|v_2^{\varepsilon_n}\right) = \underset{v_2^n \in \left(v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n\right)}{ess\ \inf} F\left(\underline{b_1}^{-1}\left(b_2\left(v_2^n\right)\right)|v_2^n\right)$$

and $\bar{v}_2^{\varepsilon_n} \in \left(v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n\right)$ such that $F\left(\underline{b_1}^{-1}\left(b_2\left(\bar{v}_2^{\varepsilon_n}\right)\right)|\bar{v}_2^{\varepsilon_n}\right) > \check{F}\left(\underline{b_1}^{-1}\left(b_2\left(v_2^{\varepsilon_n}\right)\right)|v_2^{\varepsilon_n}\right)$.[4] Since b_1, b_2 are monotone increasing, we know that $\underline{b_1}^{-1}\left(b_2\left(\cdot\right)\right)$ is monotone increasing. Letting $n \to \infty$, $\bar{v}_2^{\varepsilon_n}$ and $\underline{b_1}^{-1}\left(b_2\left(\bar{v}_2^{\varepsilon_n}\right)\right)$ both go to 0. Noting that $f\left(v_2\right) = \int_{v_2}^{\infty} f\left(v_1, v_2\right) dv_1$ is bounded below on $\left[0, \varepsilon_{n^*}\right]$ by some $\gamma > 0$, we see that we can find arbitrarily small $\bar{v}_2^{\varepsilon_n}$ such that the conditional distribution $f\left(v_1|\bar{v}_2^{\varepsilon_n}\right)$ must be arbitrarily large near $v_1 = 0$. This would violate the boundedness of the density $f\left(v_1, v_2\right)$ near $(0,0)$, and we conclude that there must exist an $n' > n^*$ such that $F\left(\underline{b_1}^{-1}\left(b_2\left(\bar{v}_2^{\varepsilon_{n'}}\right)\right)|\bar{v}_2^{\varepsilon_{n'}}\right) < \frac{1}{2}$. Given our definition of $\bar{v}_2^{\varepsilon_n}$, there exists a set, E, with positive probability measure such that agent 2 strictly prefers to bid zero on E than to bid his $b_2\left(v_2\right)$ on E. We conclude that this cannot be an equilibrium and eliminate any strategies such that for every $\varepsilon > 0$ there is a $v_2^{\varepsilon} \in (0, \varepsilon)$ where $b_2\left(v_2^{\varepsilon}\right) > v_2^{\varepsilon}$.

We are left to consider possibility (i) there exists an $\varepsilon > 0$ such that $b_2\left(v_2\right) \leq v_2$ on $[0, \varepsilon]$. Suppose $b_2\left(v_2\right) \leq v_2$ on $[0, \varepsilon]$. By Claim 4, $b_1\left(v_1\right) \geq b_2\left(v_1\right)$ for all $v_1 \in (0, \varepsilon]$. Note also that if $b_2\left(v\right) = b_2\left(v_1\right)$ for all $v \in \left[v_1 - \varepsilon', v_1\right]$ for some small $\varepsilon' > 0$, then $b_2\left(v_1\right) < v_1$ and agent 1 strictly prefers to bid $b_1\left(v_1\right) > b_2\left(v_1\right)$ than to bid $b_1\left(v_1\right) = b_2\left(v_1\right)$.[5] Given v_2 in $(0, \varepsilon]$, if $b_2\left(v_2\right) > 0$, then agent 2 pays $b_2\left(v_2\right)$ with probabilty one and never receives the prize. He strictly prefers to bid $b_2\left(v_2\right) = 0$ on $(0, \varepsilon]$, and we eliminate any strategy for agent 2 such that $v_2 \geq b_2\left(v_2\right) > 0$ for some v_2 in $(0, \varepsilon]$.

Now the only equilibrium candidates remaining are such that $b_2\left(v_2\right) = 0$ on $[0, \varepsilon]$. By another application of Claim 1, $b_1\left(v_1\right) > 0$ for $v_1 \geq \varepsilon$, and agent 2 − on the margin − never wants to increase his bid. Therefore, agent 2 strictly prefers to bid $b_2\left(v_2\right) = 0$ than to bid $b_2\left(v_2\right) > 0$ on $\mathbb{R}_+$, as desired. ■

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[4] If $F\left(\underline{b_1}^{-1}\left(b_2\left(v_2\right)\right)|v_2\right) = \check{F}\left(\underline{b_1}^{-1}\left(b_2\left(v_2^{\varepsilon_n}\right)\right)|v_2^{\varepsilon_n}\right)$ for all $v_2 \in \left(v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n\right)$, then let $\bar{v}_2^{\varepsilon_n}$ be any $v_2 \in \left(v_2^{\varepsilon_n} - \delta_n, v_2^{\varepsilon_n} + \delta_n\right)$.

[5] That is, if b_2 is constant in some interval immediately below v_1, then (i) $b_2(v_1) < v_1$ and (ii) agent 1 strictly prefers to bid $b_1(v_1) > b_2(v_1)$ than to bid $b_1(v_1) = b_2(v_1)$.

Reference

Perry, M., and Reny, P.J. (1999). "A General Solution to King Solomon's Dilemma," *Games Econom. Behavior* **26**, 279-285.

Appendix

Proofs of Claims 1−4:

Claim 1: $b_1(v_1) > 0$ for any $v_1 > 0$ and any $b_2(\cdot)$.

Proof of Claim 1: This is immediate since bidding $b_1(v_1) = v_1$ gives agent 1 strictly positive expected value for any $v_1 > 0$ and any $b_2(\cdot)$. ■

Claim 2: If $b_2(0) = 0$ and $b_2(v_1) > v_1$ then $v_1 \geq b_1(v_1)$.

Proof of Claim 2: Suppose $b_2(v_1) > v_1$.

The expected value to agent 1 from bidding $b_1(v_1) = v_1$ is

$$\int_0^{\underline{b_2^{-1}}(v_1)} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 + \int_{\overline{b_2^{-1}}(v_1)}^{v_1} v_1 \cdot f(v_2|v_1)\, dv_2.$$

Consider $b_1(v_1) > v_1$ where $b_1(v_1) > b_2(v_1)$. The expected value to agent 1 from bidding $b_1(v_1) > v_1$ is

$$\begin{aligned} & \int_0^{\underline{b_2^{-1}}(v_1)} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 \\ < & \int_0^{\underline{b_2^{-1}}(v_1)} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 + \int_{\overline{b_2^{-1}}(v_1)}^{v_1} v_1 \cdot f(v_2|v_1)\, dv_2. \end{aligned}$$

Thus, agent 1 strictly prefers to bid $b_1(v_1) = v_1$ than $b_1(v_1) > v_1$ where $b_1(v_1) > b_2(v_1)$.

Now consider $b_1(v_1) > v_1$ where $b_1(v_1) \leq b_2(v_1)$. The expected value to agent 1 from bidding $b_1(v_1) > v_1$ is now

$$\left\{ \int_{0}^{\underline{b_2^{-1}}(v_1)} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 \right.$$

$$\left. + \int_{\underline{b_2^{-1}}(b_1(v_1))}^{\overline{b_2^{-1}}(b_1(v_1))} \frac{1}{2}(v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 + \int_{\overline{b_2^{-1}}(b_1(v_1))}^{v_1} v_1 \cdot f(v_2|v_1)\, dv_2 \right\}$$

$$= \left\{ \int_{0}^{\underline{b_2^{-1}}(v_1)} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 \right.$$

$$\left. + \int_{\underline{b_2^{-1}}(b_1(v_1))}^{\overline{b_2^{-1}}(b_1(v_1))} \frac{1}{2}(v_1 - b_1(v_1)) \cdot f(v_2|v_1)\, dv_2 + \int_{\overline{b_2^{-1}}(b_1(v_1))}^{v_1} v_1 \cdot f(v_2|v_1)\, dv_2 \right\}$$

$$\leq \int_{0}^{\underline{b_2^{-1}}(v_1)} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 + \int_{\overline{b_2^{-1}}(b_1(v_1))}^{v_1} v_1 \cdot f(v_2|v_1)\, dv_2$$

$$< \int_{0}^{\underline{b_2^{-1}}(v_1)} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 + \int_{\overline{b_2^{-1}}(v_1)}^{v_1} v_1 \cdot f(v_2|v_1)\, dv_2.$$

Thus, agent 1 strictly prefers to bid $b_1(v_1) = v_1$ than $b_1(v_1) > v_1$ where $b_1(v_1) \leq b_2(v_1)$. Given that agent 1 strictly prefers to bid $b_1(v_1) = v_1$ than $b_1(v_1) > v_1$ when $b_1(v_1) \leq b_2(v_1)$ or when $b_1(v_1) > b_2(v_1)$, Claim 2 is proven. ■

Claim 3: No equilibrium exists such that $b_2(0) > 0$.

Proof of Claim 3: Suppose that $b_2(0) > 0$. First we show that for $v_1 \in (0, b_2(b_2(0)))$ we have that $v_1 \leq b_1(v_1) < b_2(v_1)$. If $b_1(v_1) = v_1$, then agent 1's expected value is v_1. If $v_1 > b_1(v_1)$, then agent 1's expected value is $\int_{0}^{b_1(v_1)} v_1 f(v_2|v_1) dv_2 - \int_{b_1(v_1)}^{v_1} b_1(v_1) f(v_2|v_1) dv_2 < v_1$. If $b_1(v_1) \geq b_2(v_1)$, then agent 1's expected value is $\int\limits_{b_2^{-1}(b_1(v_1))}^{v_1} \frac{1}{2}\left(v_1 - b_1\left(v_1\right)\right) f(v_2|v_1)dv_2 \leq 0 < v_1$, and we have the result.

Given $v_2 \in \left(0, b_2\left(b_2\left(0\right)\right)\right)$ we now know that if agent 2 wins he quits, and if he loses he pays $b_2\left(v_2\right) > 0$. It suffices to show that $b_1\left(v_1\right)$ is not bounded above by $b_2\left(0\right)$. This is will ensure that for every v_2 in some neighborhood of $v_2 = 0$, there is a strictly positive probability that agent 2 loses.

By Claim 2 above, $b_1\left(v_1\right) \leq v_1$ for all $v_1 \in \left(0, b_2\left(0\right)\right)$. Given $v_2 \in \left[0, b_2\left(0\right)\right)$, if agent 2 wins he quits and if he loses he pays at least $b_2\left(0\right)$ with certainty (since agent 1 will not quit). Thus it suffices to show that $b_1\left(v_1\right)$ is not bounded above by $b_2\left(0\right)$. This will give agent 2 strictly positive probability of losing and paying at least $b_2\left(0\right)$.[6]

If $b_1\left(v_1\right) < b_2\left(0\right)$ for all v_1, then agent 1 loses with certainty and strictly prefers to bid $b_1\left(v_1\right) = v_1$.

If $b_1\left(v_1\right) \leq b_2\left(0\right)$ for all v_1, then there must exist v_1^* such that $b_1\left(v_1\right) = b_2\left(0\right)$ for all $v_1 > v_1^*$. If $b_2\left(v_1\right) > b_2\left(0\right)$ for all $v_1 > 0$, then agent 1's expected value is zero and he strictly prefers to bid $b_1\left(v_1\right) = v_1$. If $b_2\left(v_1\right) = b_2\left(0\right)$ on precisely $[0, v]$ for $v > 0$, then for $v_1 > v_1^*$ agent 1 strictly prefers to move his bid up just slightly to win when v_2 is in $[0, v]$. Therefore, $b_1\left(v_1\right)$ cannot be not bounded above by $b_2\left(0\right)$, and Claim 3 is proven. ■

Claim 4: If $b_2\left(v_2\right) \leq v_2$ on $[0, v_1]$, then $b_1\left(v_1\right) \geq b_2\left(v_1\right)$.

Proof of Claim 4: Suppose $b_2\left(v_2\right) \leq v_2$ on $[0, v_1]$.

The expected value to agent 1 from bidding $b_1\left(v_1\right) \geq b_2\left(v_1\right)$ is then

$$\int\limits_{0}^{\underline{b_2^{-1}}(b_1(v_1))} \left(v_1 - b_2\left(v_2\right)\right) \cdot f\left(v_2|v_1\right) dv_2 + \int\limits_{\underline{b_2^{-1}}(b_1(v_1))}^{v_1} \frac{1}{2}\left(v_1 - b_2\left(v_2\right)\right) \cdot f\left(v_2|v_1\right) dv_2$$

The expected value to agent 1 from bidding $b_1\left(v_1\right) < b_2\left(v_1\right)$ is then

$$\left\{ \int\limits_{0}^{\underline{b_2^{-1}}(b_1(v_1))} \left(v_1 - b_2\left(v_2\right)\right) \cdot f\left(v_2|v_1\right) dv_2 \right.$$

$$\left. + \int\limits_{\underline{b_2^{-1}}(b_1(v_1))}^{\overline{b_2^{-1}}(b_1(v_1))} \frac{1}{2}\left(v_1 - b_2\left(v_2\right)\right) \cdot f\left(v_2|v_1\right) dv_2 - \int\limits_{\overline{b_2^{-1}}(b_1(v_1))}^{v_1} b_1\left(v_1\right) \cdot f\left(v_2|v_1\right) dv_2 \right\}$$

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[6] If $b_1\left(v_1\right)$ is not bounded above by $b_2\left(0\right)$, then for v_2 in some small neighborhood of $v_2 = 0$, agent 2 also loses with strictly positive probability.

$$< \int_{0}^{\underline{b_2^{-1}}(b_1(v_1))} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2 + \int_{\underline{b_2^{-1}}(b_1(v_1))}^{v_1} \frac{1}{2} (v_1 - b_2(v_2)) \cdot f(v_2|v_1)\, dv_2$$

Thus, agent 1 strictly prefers to bid $b_1(v_1) \geq b_2(v_1)$ than $b_1(v_1) < b_2(v_1)$ when $b_2(v_2) \leq v_2$ on $[0, v_1]$, and Claim 4 is proven.■